

ISOMETRIC EMBEDDABILITY OF SNOWFLAKES

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ABSTRACT. We show that a snowflake of a metric space with positive Hausdorff dimension does not admit an isometric embedding into euclidean space.

1. INTRODUCTION

Let (X, d) be a metric space. By euclidean space we mean \mathbb{R}^k equipped with the standard euclidean metric. Given $\lambda > 0$ a map $f : X \rightarrow \mathbb{R}^k$ gives a λ -bilipschitz embedding of (X, d) into euclidean space if:

$$\frac{1}{\lambda}d(x, x') \leq \|f(x) - f(x')\| \leq \lambda d(x, x') \quad \text{for all } x, x' \in X.$$

The map f is a bilipschitz embedding if it is a λ -bilipschitz embedding for some $\lambda > 0$. Bilipschitz embeddings are injective, so the term “embedding” is justified. The map f is an isometric embedding if it is a 1-bilipschitz embedding. Given $0 < r < 1$, the r -snowflake of (X, d) is the metric space (X, d^r) . We say that (X, d^r) is a snowflake of (X, d) . Let $K > 0$, we say that (X, d) is K -doubling if every open ball of radius t contains at most K pairwise disjoint open balls of radius $\frac{1}{2}t$. The metric space (X, d) is said to be doubling if it is K -doubling for some $K > 0$. It is easy to see that the following facts hold:

- Euclidean space is doubling.
- Any metric space which admits a bilipschitz embedding into euclidean space is doubling.
- A snowflake of a doubling metric space is doubling.

The following marvelous theorem, due to Assouad, gives a kind of converse to the simple facts listed above:

Theorem 1.1 (Assouad). *Suppose that (X, d) is doubling and $0 < r < 1$. Then the r -snowflake of (X, d) admits a bilipschitz embedding into some euclidean space.*

It is natural to wonder when snowflakes admit isometric embeddings into euclidean space. In this paper we show that this is generally not the case:

Theorem 1.2. *Suppose that (X, d) has positive Hausdorff dimension and $0 < r < 1$. Then the r -snowflake of (X, d) does not admit an isometric embedding into euclidean space.*

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2. PRELIMINARIES

Our proof depends on some basic geometric facts which we gather in this section. We begin with an elementary lemma:

Lemma 2.1. *Let $x_1, \dots, x_{l+1} \in \mathbb{R}^l$ be in general position. Then the map $\sigma : \mathbb{R}^l \rightarrow \mathbb{R}^{l+1}$ given by*

$$\sigma(y) = (\|y - x_1\|, \dots, \|y - x_{l+1}\|)$$

is injective.

Proof. We suppose otherwise towards a contradiction. Suppose that $y, y' \in \mathbb{R}^l$ are such that $y \neq y'$ and $\sigma(y) = \sigma(y')$. Let H be the set of $x \in \mathbb{R}^l$ such that $\|x - y\| = \|x - y'\|$. So $x_1, \dots, x_{l+1} \in H$. However, as H is a hyperplane of codimension one, this implies that x_1, \dots, x_{l+1} are not in general position. \square

We let $D \subseteq \mathbb{R}^{l+1}$ be the image of σ and let $\tau : D \rightarrow \mathbb{R}^l$ be the compositional inverse of σ . That is, if $\bar{t} = (t_1, \dots, t_{l+1}) \in D$ then $\tau(\bar{t})$ is the unique $y \in \mathbb{R}^l$ such that:

$$\|y - x_i\| = t_i \quad \text{for all } 1 \leq i \leq l+1.$$

We make use of the following:

Lemma 2.2. *There are smooth submanifolds $D_1, \dots, D_m \subseteq \mathbb{R}^{l+1}$ such that $D = D_1 \cup \dots \cup D_m$ and the restriction of τ to each D_i is smooth.*

Proof. It is presumably easy to prove Lemma 2.2 in an elementary way. However, the present author is a logician. Therefore, we give a very general proof using semialgebraic geometry. A set $A \subseteq \mathbb{R}^k$ is semialgebraic if it is a finite union of sets of the form

$$\{\bar{x} \in \mathbb{R}^k : p(\bar{x}) \geq 0\} \quad \text{for polynomial } p.$$

A function $f : A \rightarrow B$ between semialgebraic subsets $A, B \subseteq \mathbb{R}^k$ is semialgebraic if its graph is a semialgebraic subset of $\mathbb{R}^k \times \mathbb{R}^k$. We refer to [BCR87] for information about semialgebraic geometry. It is well known that every semialgebraic subset of euclidean space is a finite union of smooth submanifolds of euclidean space and if $A \subseteq \mathbb{R}^k$ and $f : A \rightarrow \mathbb{R}^n$ are semialgebraic then A can be written as a finite union of smooth submanifolds of \mathbb{R}^k in such a way that the restriction of f to every submanifold is smooth. It is an immediate consequence of Tarski-Seidenberg quantifier elimination that $D \subseteq \mathbb{R}^{l+1}$ and $\tau : D \rightarrow \mathbb{R}^l$ are both semialgebraic. Lemma 2.2 follows. \square

Lemma 2.3. *Let $A \subseteq D$. The Hausdorff dimension of $\tau(A)$ is no greater than the Hausdorff dimension of A .*

Lemma 2.3 is a straightforward consequence of Lemma 2.2 and a few standard facts about Hausdorff dimension which can be found in [Mat95] or other places. We let \dim be the Hausdorff dimension.

Proof. Let D_1, \dots, D_m be as in the statement of Lemma 2.2. Then:

$$\dim(A) = \max\{\dim(D_i \cap A) : 1 \leq i \leq m\}$$

and

$$\dim(\tau(A)) = \max\{\dim(\tau(D_i \cap A)) : 1 \leq i \leq m\}.$$

Smooth maps do not raise Hausdorff dimension, therefore:

$$\dim(\tau(D_i \cap A)) \leq \dim(D_i \cap A) \quad \text{for all } 1 \leq i \leq m.$$

\square

3. PROOF

In this section we prove Theorem 1.2. We let (X, d) be a metric space with positive Hausdorff dimension and $0 < r < 1$. Let D and τ be as in the previous section and let \dim be the Hausdorff dimension. We suppose toward a contradiction that $\iota : X \rightarrow \mathbb{R}^l$ gives an isometric embedding of (X, d^r) into euclidean space. We may suppose that $\iota(X)$ contains $l + 1$ points y_1, \dots, y_{l+1} in general position. If this is not the case then $\iota(X)$ is contained in a hyperplane with positive codimension, and we replace ι with an isometric embedding into a euclidean space with smaller dimension. We let $x_1, \dots, x_{l+1} \in X$ be such that

$$\iota(x_i) = y_i \quad \text{for all } 1 \leq i \leq l + 1.$$

It follows from Lemma 2.1 that for all $x \in X$, $\iota(x)$ is the unique $y \in \mathbb{R}^l$ such that

$$\|y_i - y\| = d(x_i, x)^r \quad \text{for all } 1 \leq i \leq l + 1.$$

Let $X' = X \setminus \{x_1, \dots, x_{l+1}\}$. Let U be the open subset of \mathbb{R}^{l+1} consisting of elements with positive coordinates. Let $f : X' \rightarrow U$ be given by

$$f(x) = (d(x_1, x), \dots, d(x_{l+1}, x))$$

and $g : U \rightarrow U$ be given by

$$g(t_1, \dots, t_{l+1}) = (t_1^r, \dots, t_{l+1}^r)$$

Note that $g \circ f$ maps X' into D . The restriction of ι to X' can be factored as the composition

$$X' \xrightarrow{f} U \xrightarrow{g} U \xrightarrow{\tau} \mathbb{R}^l.$$

As f gives a lipschitz map $(X', d) \rightarrow U$ we have $\dim f(X') \leq \dim(X', d)$. As g is smooth it does not raise Hausdorff dimension so $\dim(g \circ f)(X') \leq \dim(X', d)$ as well. It follows from Lemma 2.3 that $\dim(\tau \circ g \circ f)(X') \leq \dim(X', d)$. Therefore $\dim \iota(X') \leq \dim(X', d)$. As $X \setminus X'$ is finite we have $\dim \iota(X) \leq \dim(X, d)$. As $\iota(X)$ is isometric to (X, d^r) this implies that $\dim(X, d^r) \leq \dim(X, d)$. However, it follows immediately from the definition of Hausdorff dimension that $\dim(X, d^r) = \frac{1}{r} \dim(X, d)$. This yields a contradiction as $\frac{1}{r} > 1$ and $\dim(X, d) > 0$.

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